# The Algebra of Geometry

*Cartesian, Areal and Projective Co-ordinates*

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# Introduction

This is a book about points, lines, triangles and conics situated in a plane. The circle, being a special conic in the Euclidean plane, is given due prominence. In fact Chapter 6 is entirely devoted to the circle and to the properties of cyclic quadrilaterals. Before that I attempt to describe comprehensively the standard properties of triangles, quadrangles, hexagons, circles and conics. In the later chapters, the amount of material competing for possible inclusion is so immense that choices have had to be made.

I decided to give no systematic account of triangle centres, which is a modern and competitive industry, now involving over three thousand points. I introduce a limited number of such centres, but only as they occur naturally in topics under discussion. A second choice is not to discuss in any detail the underlying groups of transformations that distinguish Euclidean, affine and projective geometry. These groups are of course mentioned, but as they have been the centre of attention in so many books during the twentieth century, it seems to me that one can be sure that readers are now well acquainted with the theoretical foundations of the subject, or, if not, that they have an abundance of splendid material to which they can refer. A third choice is to include nothing about cubic curves or curves of even higher degree. Some of these curves do, however, make an appearance in the examples and exercises.

What I have chosen to include has been dictated to a large extent by two major considerations. These are, first and foremost, what the book is attempting to provide for readers. In short that is an account of the use of co-ordinates in Euclidean, affine and projective geometry, with the aim of enabling others to use such techniques efficiently. A second aim was to incorporate some of my main interests over the last fifteen years.

Since I started to study geometry I have not become aware of any books written in English in recent years specialising in co-ordinate geometry. Sometimes one gets the impression that mathematicians regard those who practise the arts of synthetic geometry with greater esteem than those who profess skill with co-ordinate methods. To be fair, methods of studying geometry have swung in and out of fashion. So I prefer the point of view expressed by Maxwell, who wrote in [25], 'In a good geometrical technique the methods of Pure Geometry and Analytical Geometry should go hand in hand, each helping the other forward, and weakness in either will be in danger of leading to weakness in both'. However, I do not think I need to defend my point of view. It is a fact that as far as I am aware, there are no good modern sources written with the specific intention of describing co-ordinate methods and this has been most persuasive in framing the decision of what I should write about. The book covers the use of Cartesian co-ordinates, areal co-ordinates and homogeneous projective co-ordinates and describes the main properties and applications, as appropriate, of each of these disciplines. A further decision had to be made. The temptation is always to write an introductory text, but elementary geometrical texts are in fact available that deal with both pure and analytic methods, such as the Open University Course by Brannan, Esplen and Gray [12]. So, quite deliberately, I have written a book that starts by quoting a large number of introductory results in its first two chapters, with the intention of leaving room in the text for more advanced applications, illustrating to greater advantage the power of co-ordinate methods. Some of these preliminary results I first heard of as a boy at Rossall School, Fleetwood, and I have been reminded of some of them by looking again at Carr's Synopsis [13].

However, the book is not exclusively about co-ordinates. It would be preposterous to neglect synthetic methods, particularly in dealing with topics such as circles. It would also seem absurd to exclude complex variable methods and polar reciprocation, if readers are to be given a reasonably comprehensive account of the properties of triangles, circles and conics. Chapters 14 and 15 respectively provide some account of these topics.

The secondary reasons dictating the choice of contents are concerned with what I have been doing these last fifteen years besides teaching at Clifton College, Bristol. Thus certain of the chapters in the book contain an account of my own work, particularly research in collaboration with Dr Geoff Smith of the University of Bath. An attempt has been made to write an alternative and more co-ordinate based account of this work and my thanks to him for his support and his patience with some of my more involved calculations. My other contribution has been in the training of UK teams for the International Mathematical Olympiad and it has been an immense privilege to be involved with so many brilliant students. Though co-ordinate methods are regarded as non-essential by the IMO authorities and indeed great effort is made in the composition of the examinations to ensure that geometrical problems are not easily treated by co-ordinate methods, good pupils are not to be denied knowledge that may be useful to them. Thus the emphasis in the text on problem solving derives from IMO activities. The text, in fact, includes dozens of worked examples and over eight hundred problems, many of them original. Some of them have been composed by Dr David Monk [28], formerly of the University of Edinburgh, with whom I have maintained an active correspondence over the years, and who has been a great influence. I thank him for allowing me to publish about twenty of his original problems and for his comments on the material of several chapters. About thirty problems in the book are taken from Wolstenholme's famous book of problems [37], first published in the middle of the 19th century. These are labelled in the text with the initials JW. One problem was devised by Dr Gerry Leversha of St Paul's School, London and two by Prof Ben Green of the University of Cambridge. These are labelled in the text by their initals GL and BG respectively.

The book contains a certain amount of original material, not previously published. I mention, in this context, part of Chapter 8 and the whole of Chapter 12, though one of the main theorems in Chapter 12 is due to Vin de Silva [16]. Until very recently Chapter 17 fell into this category and is perhaps the chapter that has given me the greatest pleasure to devise. This is partly because of the way it has enabled me to draw generalisations of Euclidean results in the projective plane. The book is illustrated with figures drawn using the geometrical software CABRI. But drawing diagrams in books is not the main use of geometrical software. Many of the recent developments in geometry have been made possible by the use of such software, with its highly accurate numerical background, allowing vast numbers of complicated constructions and numerical tests to be carried out in a single day. Such software allows conclusive checks to be carried out, for example, on whether three points are collinear or three lines are concurrent or parallel, and this greatly facilitates the framing of hypotheses. In fact geometry software is one of the main reasons for the revival of interest in the subject over the past ten years.

I confess that the book has not been written with any undergraduate university course in mind. In fact it is not written for a particular student audience. It is written for anyone interested in geometry, of whatever age and wherever they may be studying, teaching or working. I am delighted therefore to have the backing of the present publishers, and thank them for the care that has been taken with the final manuscript and the final product.

Christopher Bradley, Bristol, UK, May 2007.

# Publisher's remarks

In the 19th century the algebra of the plane was common currency among mathematicians. A hundred years later the main fronts of mathematical research have moved elsewhere. However, the recent ready availability of computer graphics packages make it possible for a skilled enthusiast to discover new empirical truths about geometrical configurations with great rapidity and confidence. However, such conjectures must be proved, else we have no idea why they are true.

Complicated geometrical configurations are often not amenable to synthetic proofs (the combining of previously known results in a cunning way). Sometimes one must calculate, and there is nothing wrong with that. If you are going to calculate, then of course you must calculate well. Christopher Bradley's dexterity with the algebra of the plane is marvellous to behold.

$\mathtt{HP}^{\mathtt{n}}$, 2007.

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